JOURNAL OF NUMBER THEORY 26, 179–191 (1987)

The Distribution of the Number of Factors in a Factorization

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Communicated by H. Halberstam

Received May 5, 1986

A factorization of a positive integer n, here, is a specification of $m(d)$, the power to which d occurs in $\prod d^{m(d)}=n$; order is immaterial. The number of factors in a factorization has two natural interpretations: as $\sum m(d)$ or as the number of non-zero $m(d)$, that is, counting or not counting multiplicity. In either case, the factorizations of positive integers $\leqslant x$ into k factors number approximately $\psi_v(u)\times(\log x)^{k-1}/k!(k-1)!$, where $u=k(k-1)/\log x$, and ψ_v is either $\Gamma(2-u)$ or $1/\Gamma(1+u)$ according to whether multiplicity is counted or not. In the former case, we must have $u\leqslant 2-\varepsilon$; in the latter, $u\leqslant C$. © 1987 Academic Press, Inc.

1. INTRODUCTION

A factorization such as $48=3\cdot4\cdot4$ may be identified with the multiplicity function m: $\{2, 3, 4,...\}\to\{0, 1, 2,...\}$, which specifies the power to which each factor occurs. In the example, $m(3)=1$, $m(4)=2$ and $m(n)=0$ otherwise.

Let F denote the set of all m: $\{2, 3, 4,...\}\to\{0, 1, 2,...\}$ such that $m(n)$ differs from zero for only finitely many n. Let $|m|=\prod_{d=2}^{\infty}d^{m(d)}$ denote the number of which m is a factorization, let $\Omega(m)=\sum_{d=2}^{\infty}m(d)$, and let $\omega(m)=\#\{d\colon m(d)>0\}$. We estimate two counts of factorizations using k factors:

$$N_\Omega(x,k):=\#\{m\in F\colon |m|\leqslant x \text{ and } \Omega(m)=k\}, \tag{1.1}$$

and

$$N_\omega(x,k):=\#\{m\in F\colon |m|\leqslant x \text{ and } \omega(m)=k\}.$$

THEOREM. *Let $u=u(x,k)=k(k-1)/\log x$. Then, for all ε, $0<\varepsilon<2$, uniformly in $u\leqslant 2-\varepsilon$,*

(i) $N_\Omega(x,k)=(\Gamma(2-u)\,x(\log x)^{k-1}/k!(k-1)!)(1+O_\varepsilon(\sqrt{k/\log x}))$

0022-314X/87 $3.00

while for all $C>0$, uniformly in $u \leqslant C$,

$$\text{(ii)} \quad N_\omega(x, k) = (x(\log x)^{k-1}/k!(k-1)!\,\Gamma(1+u))(1+O_C(\sqrt{k/\log x})).$$

Minor modifications in the proof of (ii) give a similar result for the number of factorizations into distinct factors, that is, with $m(d) \leqslant 1$ for all d. The only change in the estimate is that $\Gamma(1+u)$ is replaced with $\Gamma(2+u)$.

It is easily verified that the estimates (i) and (ii) are, for a given x, largest for those k near $\sqrt{\log x}$, and that most of the factoriztions counted have $(1+o(1))\sqrt{\log x}$ factors. This puts u close to 1, and then $(1/\Gamma(1+u))/(1/\Gamma(2+u))$ is close to 1/2. Thus about half the factorizations are repetition free, as we should expect since $\prod_{n=2}^{\infty}(1-1/n^2) = \frac{1}{2}$. This is not to say that most numbers $\leqslant x$ have so many as $\sqrt{\log x}$ factors. Rather, a few highly composite numbers generated most of the factorizations, and for these "round" numbers most factorizations involve about $\sqrt{\log x}$ factors.

The study of factorizations goes back at least to MacMahon, who introduced a generating function

$$f(s) := \prod_{n=2}^{\infty} (1+1/n^s) = \sum_{m \in F} |m|^{-s}, \tag{1.2}$$

and to Oppenheim, who used it to prove

$$\#\{m: |m| \leqslant x\} = (1+O(1/\log x))\, xe^{2\sqrt{\log x}}/2\sqrt{\pi}(\log x)^{3/4}. \tag{1.3}$$

(Oppenheim gave also an asymptotic expansion, good to within a factor of $1+O((\log x)^{-n})$ [5].) His proof was a blend of combinatorics and complex analysis, in which the Riemann zeta function and Bessel functions figured prominently.

Later, Sathe estimated $\#\{n: 1 \leqslant n \leqslant x \text{ and } w(n)=k\}$ and its Ω counterpart [6]. His inductive argument was soon superseded by the analytic approach of Selberg, who began with a consideration of $(\zeta(s))^z$ and its relation to $\sum_{n=1}^{\infty} z^{\omega(n)}/n$. This gave simpler proofs and wider generality [7]. There are also some results on the number of factorizations m with $|m| \leqslant x$ employing either only factors $\leqslant y$, or factors $> y$ [2, 3].

Here we apply the approaches of Oppenheim and Selberg, and begin with the definitions

$$\begin{aligned} f_\Omega(s, z) &:= \prod_{n=2}^{\infty} (1-z/n^s)^{-1}, \\ f_\omega(s, z) &:= \prod_{n=2}^{\infty} (1+z/(n^s-1)). \end{aligned} \tag{1.4}$$

For much of the proof of the theorem, the same argument works for either case, and we will write simply $f(s,z)$. Further notation which does not distinguish Ω from ω should also be understood to mean that either one will work. Finally, let v denote either of ω or Ω. Then

$$f(s, z) = \sum_{m \in F} z^{v(m)} |m|^{-s}, \tag{1.5}$$

the sum and product being uniformly and absolutely convergent to $f(s, z)$ on all sets of the forms $(|z| \leqslant C) \times (\mathrm{Re}(s) \geqslant 1 + \varepsilon)$, provided $C < 2$ in the case of $v = \Omega$.

Now $f(s, z)$ is a complex analytic function in two variables, and we refer the reader to [4], and in particular to Theorems 2.21 and 2.26, for the justification of the use we will make of Cauchy's theorem and the like in this context.

Let

$$D(x, z) := \sum_{|m| < x} (x - |m|)\, z^{v(m)}. \tag{1.6}$$

Then by Perron's formula,

$$D(x, z) = \frac{1}{2\pi i} \int_{2 - i\infty}^{2 + i\infty} \frac{x^{s+1}}{s(s+1)} f(s, z)\, ds \tag{1.7}$$

(provided $|z| < 2$ in case $v = \Omega$), and so

$$M(x, k) := \sum_{1 \leqslant n \leqslant x} (x - n)\, N(n, k) = \frac{1}{2\pi i} \oint_{|z| = 1} z^{-k-1} D(x, z)\, dz. \tag{1.8}$$

Now the idea is to first estimate $M(x, k)$ by way of (7) and (8), and then interpolate to recover $N(x, k)$.

2. Analysis

By writing $f(s, z) = g(s, z) \exp(z\zeta(s))$, we obtain an analytic extension of f to $(\mathrm{Re}(s) > \frac{1}{2},\ s \neq 1) \times (|z| \leqslant C)$ (subject to $C < 1/\sqrt{2}$ in case $v = \Omega$). The two functions $g(s, z)$ so defined are given by

$$g_\Omega(s, z) = e^{-z} \prod_{n=2}^{\infty} (1 - z/n^s)^{-1} \exp(-z/n^s), \tag{2.1}$$

and

$$g_\omega(s, z) = e^{-z} \prod_{n=2}^{\infty} (1 + z/(n^s - 1)) \exp(-z/n^s),$$

and the products are absolutely convergent to $g(s, z)$ uniformly on sets of the form $(\mathrm{Re}(s) \geqslant \frac{1}{2} + \varepsilon) \times (|z| \leqslant C)$, provided $C < 2^{(1/2+\varepsilon)}$ in case $v = \Omega$. Furthermore, $|g(s, z)|$ is bounded on any such set.

Since $|\zeta(s)| = O(\log t/\log\log t)$ for $s = \sigma + it$ with $\sigma \geqslant 1$ and $|t| \geqslant e^e$ [8], we can replace the contour $(2 - i\infty, 2 + i\infty)$ of (1.7) with $\mathscr{C}$, which follows the line $\sigma = 1$ from $1 - i\infty$ to $1 - i/6$, then circles to the right of 1 to $1 + i/6$, and then continues along $\sigma = 1$ to $1 + i\infty$.

It turns out that most of the integral in (1.7) comes from the circular arc. Let $\mathscr{E}$ be the path which coincides with $\mathscr{C}$ except that it bypasses 1 to the left. Let $\mathscr{D}$ be the circular path taken counterclockwise around 1 at a radius of $\frac{1}{6}$. Then

$$D(x, z) = \frac{1}{2\pi i}\left(\oint_D + \int_{\mathscr{E}} \frac{x^{s+1} f(s, z)\, ds}{s\,(s+1)}\right) \tag{2.2}$$

$= \bar{D}(x, z) + E(x, z)$, say. Now we can split $M(x, k)$ into the corresponding pieces

$$M(x, k) = \frac{1}{2\pi i} \oint_{|z|=1} z^{-k-1} (\bar{D}(x, z) + E(x, z))\, dz \tag{2.3}$$

$= \bar{M}(x, k) + E_k(x)$, say.

The main result of this section is

LEMMA 1. *$\bar{M}(x, k) = (1 + O_C(k/\log x))\, h(1, u) \cdot \frac{1}{2} x^2 (\log x)^{k-1}/k!(k-1)!$, uniformly in $u = k(k-1)/\log x \leqslant C$, where $C < 2$ in case $v = \Omega$, and C can be taken arbitrarily large in case $v = \omega$. (The definition of $h(s, z)$ lies just ahead. It turns out that $h(1, z) = \Gamma(2 - z)$ or $1/\Gamma(1 + z)$ as $v = \Omega$ or ω).*

Remark. Given this lemma, it is easy to guess the theorem: just differentiate $\bar{M}(x, k)$ and ignore the error terms.

In the proof of Lemma 1 we are only concerned with s near 1, so it is natural to write $\zeta(s) = 1/(s-1) + \gamma + \phi(s)$, where $\phi(1) = 0$ and $\phi(s)$ is entire. Next, let

$$h(s, z) = \frac{2}{s(s+1)} e^{-z/(s-1)} f(s, z). \tag{2.4}$$

Then $h(s, z) = (2/s(s+1)) \exp(\gamma z + z\phi(s))\, g(s, z)$ and is given by the series expansion

$$h(s, z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} h_{ij} (s-1)^i z^j. \tag{2.5}$$

where

$$h_{ij} = -\frac{1}{4\pi^2} \oint_{|s-1|=1/6} \oint_{|z|=1} (s-1)^{-i-1} z^{-j-1} h(s, z)\, dx\, ds.$$

Now for a time consider just the case $v = \Omega$. We may replace the above circles of integration with $|s-1| = \varepsilon$ and $|z| = 2^{1-\varepsilon} j/(j+1)$, and for small ε (say $0 < \varepsilon < 1/6$) conclude from (2.5), by estimating the integral, that

$$|h_{ij}| \ll_\varepsilon j\varepsilon^{-i} 2^{-(1-\varepsilon)j}. \tag{2.6}$$

The point of (2.6) is that later on we need bounds which are good for large j, even at the expense of bad growth in i.

On the other hand, in the case $v = \omega$, $h(s, z)$ is entire in z whenever $\mathrm{Re}(s) > \frac{1}{2}$, so again by (2.5),

$$|h_{ij}| \ll_{\varepsilon, C} (\tfrac{1}{2} - \varepsilon)^{-i} C^{-j} \tag{2.7}$$

for $0 < \varepsilon < \frac{1}{6}$ and any $C > 0$.

Now for $|z| \leqslant 1$, for either $v = \Omega$ or $v = \omega$,

$$\bar{D}(x, z) = \frac{1}{2} x^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} h_{ij} \left(\frac{1}{2\pi i}\right) \oint_{\mathscr{S}} x^{s-1} e^{z/(s-1)} (s-1)^i z^j\, ds, \tag{2.8}$$

since the absolute and uniform convergence of the double sum of (2.5) to $h(s, z)$ on $(|z| \leqslant 1) \times (|s-1| \leqslant \frac{1}{6})$ permits the interchange of sums and integral.

The integrals $\oint_{\mathscr{S}} x^{s-1} e^{z/(s-1)} (s-1)^i\, ds$ are connected with Bessel functions. Here all we need is the series expansion

$$\frac{1}{2\pi i} \oint_{\mathscr{S}} x^{s-1} e^{z/(s-1)} (s-1)^i\, ds = \sum_{l=0}^{\infty} \frac{z^{i+l+1} (\log x)^l}{l!\,(i+l+1)!}, \tag{2.9}$$

which may as easily be derived from the series expansion of $e^{z/(s-1)}$ as looked up in one of the standard references. Putting this into (2.8) gives

$$\begin{aligned} \bar{D}(x, z) &= \frac{1}{2} x^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} h_{ij} \sum_{l=0}^{\infty} \frac{z^{i+j+l+1} (\log x)^l}{l!\,(i+l+1)!} \\ &= \frac{1}{2} x^2 \sum_{k=1}^{\infty} z^k (\log x)^{k-1} \sum_{r=0}^{k-1} (\log x)^{-r} \sum_{j=0}^{r} h_{r-j,j} \\ &\quad \times ((k-1-r)!\,(k-j)!)^{-1} \end{aligned} \tag{2.10}$$

so that

$$\bar{M}(x, k) = \frac{1}{2} x^2 (\log x)^{k-1} \sum_{r=0}^{k-1} (\log x)^{-r} \sum_{j=0}^{r} \frac{h_{r-j,j}}{(k-j)!\,(k-1-r)!}. \tag{2.11}$$

Now let

$$\bar{M}_j(x, k) = \frac{1}{2} x^2 (\log x)^{k-1} \sum_{r=j}^{k-1} (\log x)^{-r} \frac{h_{j,r}}{(k-1-r)!\,(k+j-r)!}, \tag{2.12}$$

so that

$$\bar{M}(x, k) = \sum_{j=0}^{k-1} \bar{M}_j(x, k).$$

The bulk of $\bar{M}(x, k)$ comes from $\bar{M}_0(x, k)$, which is

$$\frac{1}{2} x^2 \frac{(\log x)^{k-1}}{k!(k-1)!} \sum_{r=0}^{k-1} h_{0,r} k_r (k-1)_r / (\log x)^r, \tag{2.13}$$

where k_r denotes $k!/(k-r)!$. Since $u = k(k-1)/\log x$, this sum should be close to $\sum_{r=0}^{\infty} h_{0,r} u^r = h(1, u)$. (And this, it turns out, $= \psi_\nu(u)$.)

The errors in this approximation have two sources. First, $u^r \neq k_r(k-1)_r/(\log x)^r$. Second, $\bar{M}_j(x, k)$ for $j \geqslant 1$ have been left out.

The error in replacing $k_r(k-1)_r/(\log x)^r$ with u^r is $O(\sum_{r=[\sqrt{k}]}^{\infty} |h_{0,r}|\, u^r) + O(\sum_{r=2}^{[\sqrt{k}]} |h_{0,r}|\, (u^r - k_r(k-1)_r/(\log x)^r)$. The first piece is $\ll \sum_{r > \sqrt{k}} C^{-r} u^r \ll_\varepsilon (u/C)^{\sqrt{k}}$, uniformly in $u \leqslant C(1-\varepsilon)$, provided $C < 2$ in the case $\nu = \Omega$. Note that $(u/C)^{\sqrt{k}} \ll k/\log x$ here.

To estimate the second piece of the first error, we start with the observation that $k_r(k-1)_r/k^r(k-1)^r = 1 + O(r^2/k)$. Thus,

$$\left| \sum_{r=0}^{[\sqrt{k}]} h_{0,r}(u^r - k_r(k-1)_r/(\log x)^r \right| \ll \sum_{r=2}^{[\sqrt{k}]} (r^2/k)\, C^{-r} u^r$$
$$\ll_\varepsilon u^2/k \ll k^3/(\log x)^2 \ll k/\log x. \tag{2.14}$$

Hence,

$$\bar{M}_0(x, k) = (1 + O(k/\log x))\, h(1, u) \cdot \tfrac{1}{2} x^2 (\log x)^{k-1}/k!(k-1)!. \tag{2.15}$$

The remainder of $\bar{M}(x, k)$ is

$$\ll (x^2 (\log x)^{k-1}/k!(k-1)!) \sum_{j=1}^{k-1} (\log x)^{-j} \sum_{l=0}^{j-1} h_{j-l,l} k_l (k-1)_j, \tag{2.16}$$

and, as we shall see, the double sum in (2.16) is $\ll_\varepsilon k/\log x$ under our constraints on u. The less favorable case is $\nu = \Omega$. In this case, from (2.6) the double sum in (2.16) is $\ll_\varepsilon \sum_{j=1}^{k-1} (k-1)_j (\log x)^{-j} \sum_{l=0}^{j-1} l\varepsilon^{l-j} \cdot 2^{-(1-\varepsilon)l} k_l$.

In the inner sum here, the ratio of consecutive terms is $(1+1/l)\,\varepsilon\, 2^{-(1-\varepsilon)}(k-l)$. Thus the largest term occurs with l near k, and is $\ll$ the last term, while for $l \leqslant k - 4/\varepsilon$, the ratio of consecutive terms exceeds 2. Hence the inner sum is $\ll_\varepsilon$ (the last of its terms), which is $(j-1)\,\varepsilon^{-1} 2^{-1(-\varepsilon)(j-1)} k_{j-1} \ll_\varepsilon j 2^{-(1-\varepsilon)j} k_{j-1}$. Consequently the double sum above is $\ll \sum_{j=1}^{k-1} (k-1)_j\, k_{j-1} 2^{-1(-\varepsilon)j} (\log x)^{-j} \ll_\varepsilon k/\log x$, provided $u < 2^{1-2\varepsilon}$. This proves Lemma 1 for the case $\nu = \Omega$. The similar but simpler case of $\nu = \omega$ is left to the reader.

3. Estimation of $E_k(x)$

Again the more difficult case is with $\nu = \Omega$. We have in either case that

$$E_k(x) = \frac{1}{2\pi i} \oint_{|z|=1} z^{-k-1} E(x, z)\, dz$$

$$= \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{x^{s+1}}{s(s+1)} \oint_{|z|=1} \frac{1}{2\pi i} z^{-k-1} g(s, z) \exp(z\zeta(s))\, dz\, ds. \quad (3.1)$$

The order of integration is interchangeable since the double integral is absolutely convergent.) If we now write $g(s, z) = \sum_{n=0}^{\infty} z^n g_n(s)$, then $g_n(s) = (1/2\pi i) \oint_{|z|=1} z^{-n-1} g(s, z)\, dz$, and so

$$E_k(x) = \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{x^{s+1}}{s(s+1)} \sum_{j=0}^{k} \frac{1}{j!} \zeta(s)^j g_{k-j}(s)\, ds. \quad (3.2)$$

LEMMA 2. *For the case* $\nu = \Omega$,

$$|E_k(x)| \ll x^2 \cdot 2^{-k} \exp(O(\sqrt{k})),$$

while for $\nu = \omega$,

$$|E_k(x)| \ll x^2 \exp(-\tfrac{1}{2} k \log\log k + O(k)),$$

in both cases with no restrictions on k.

We begin the proof with some estimates of $|g(s, z)|$ and $|g_n(s)|$. First consider the case $\nu = \Omega$. In this case, $g(s, z) = e^{-z} \prod_{n=2}^{\infty} (1 - z/n^s)^{-1} e^{-z/n^s}$. On the straight portion of $\mathscr{C}$, $s = 1 + it$. Let $\lambda = z/2^s$ and $q(\lambda) = (1-\lambda)^{-1} (\lambda^3/3 - \sum_{k=3}^{\infty} \lambda^k/k(k+1))$. Then for $s = 1 + it$,

$$g(s, z) = (1 + O(z^3)) \exp(\tfrac{1}{2} z^2 (\zeta(2s) - 1))\, e^{q(\lambda)}, \quad (3.3)$$

and since $g_k(s) = (1/2\pi i) \oint_{|z|=R} z^{-k-1} g(s, z)\, dz$,

$$|g_k(s)| \ll R^{-k} \exp\left(\frac{1}{2}\left(\frac{\pi^2}{6} - 1\right) R^2\right) \int_{-\pi}^{\pi} |e^{q(\lambda)}|\, d\theta, \tag{3.4}$$

on substituting $z = Re^{i\theta}$ and $\lambda = Re^{i\theta}/2^s$, with $R < 2$. Simplifying this gives

$$|g_k(s)| \ll R^{-k} \exp(1/(2-R)), \tag{3.5}$$

since, as is readily seen, $|q(\lambda)| \leqslant (2(1-\lambda))^{-1}$ here.

If we now take $R = 2 - \sqrt{2/k}$, this becomes

$$|g_k(s)| \ll 2^{-k} \exp(O(\sqrt{k})), \tag{3.6}$$

uniformly in $k \geqslant 1$ and in $s = 1 + it$.

We now modify $\mathscr{E}$ to $\mathscr{E}'$ by taking a tighter arc to the left of 1, this time at a radius of $1/(10\sqrt{k})$. Again using the integral representation of $g_k(s)$ with $|z| = R = 2 - \sqrt{2/k}$ we get

$$|g_k(s)| \ll 2^{-k} \exp(O(\sqrt{k})) \tag{3.7}$$

on $|s-1| = 1/(10\sqrt{k})$, $\mathrm{Re}(s) \leqslant 1$.

By Cauchy's theorem,

$$E_k(x) = \frac{1}{2\pi i} \int_{\mathscr{E}'} \frac{x^{s+1}}{s(s+1)} \sum_{j=0}^{k} \frac{1}{j!} (\zeta(s))^j g_{k-j}(s)\, ds. \tag{3.8}$$

We now divide $\mathscr{E}'$ into two parts, a central segment and the wings $|t| \geqslant T$, where T is some large but as yet arbitrary constant.

Let $C(t) = \sup_{r \geqslant t}(|\zeta(1+ir)|/(2 \log\log r/\log r))$. Then $C(t)$ exists for all sufficiently large t, from [8]. Clearly $C(t)$ is non-increasing. Thus the contribution of the wings to $E_k(x)$ satisfies

$$|E_k(x)|_{\text{wings}} \ll x^2 \int_T^{\infty} (1/t^2) \sum_{j=0}^{k} (1/j!)\, C(T)^j (\log t/2 \log\log t)^j\, |g_{k-j}(1+it)|$$

$$\ll x^2 2^{-k} \int_T^{\infty} (1/t^2) \sum_{j=0}^{k} (1/j!)(2C(T))^j (\log t/\log\log t)^j\, dt, \tag{3.9}$$

since the missing 2^{-j} offsets the absent $\exp O(\sqrt{k-j})$ from (3.6). On the substitution $w = \log t$, this last bound simplifies to $\ll x^2 2^{-k} \int_{\log T}^{\infty} \exp(-w + 2wC(T)/\log w)\, dw = o(x^2 \cdot 2^{-k})$ as $T \to \infty$. Thus for T sufficiently large, but fixed,

$$|E_k(x)|_{\text{wings}} \leqslant 2^{-k} x^2. \tag{3.10}$$

On the inner segment of $\mathscr{C}'$ we further distinguish three cases: $1 \leqslant |t| \leqslant T$, $1/10\sqrt{k} \leqslant |t| \leqslant 1$, and the arc to the left of 1 at radius $1/10\sqrt{k}$. From $1 \leqslant |t| \leqslant T$ the contribution to $E_k(x)$ is

$$\ll x^2 e^{O(\sqrt{k})} \sum_{j=0}^{k} (1/j!)\, 2^{j-k} \Big(\sup_{1 \leqslant |t| \leqslant T} |\zeta(1+it)| \Big)^j \ll x^2 2^{-k} e^{O(\sqrt{k})}.$$

From $1/(10\sqrt{k}) \leqslant |t| \leqslant 1$ we get a contribution of

$$\ll x^2 e^{O(\sqrt{k})} \int_{1/10\sqrt{k}}^{1} \sum_{j=0}^{k} (1/j!)\, 2^{j-k} (1/t + O(1))^j \, dt.$$

Since $(a+b)^j \leqslant 2^j(a^j + b^j)$ for $a, b > 0$, the bound above is

$$\ll x^2 2^{-k} e^{O(\sqrt{k})} \int_{1/10\sqrt{k}}^{1} \sum_{j=0}^{k} (1/j!)\, 4^j t^{-j} \, dt \ll x^2 2^{-k} \exp(50\sqrt{k}),$$

as provided for in Lemma 2. Similar calculations on the arc $|s-1| = 1/10\sqrt{k}$ complete the proof of Lemma 2 for the case $\nu = \Omega$.

For the case $\nu = \omega$, we first note that for s on $\mathscr{C}'$ with $\sigma = 1$

$$|g(s, z)| \ll \exp(|z| \log |z| + O(|z|)), \tag{3.11}$$

uniformly for all complex z. Thus for any $R > 0$, $g_k(s) = 1/2\pi i \oint_{|z|=R} z^{-k-1} g(s, z)\, dz$, and with $R = k/\log k$, we get

$$|g_k(s)| \ll \exp(-k \log k + k \log\log k + O(k)), \tag{3.12}$$

uniformly for $s = 1 + it$ with $|t| \geqslant 1/10\sqrt{k}$.

On the arc $\mathrm{Re}(s) < 1$, $|s-1| = 1/10\sqrt{k}$, and

$$|g(s, z)| \ll \exp((1-\sigma)^{-1}(|z|^{(1-\sigma)/\sigma} - 1) + O(|z|)), \tag{3.13}$$

an estimate which follows without difficulty from the definition (2.1).

Now taking $R = k^\sigma$ in the integral for $g_k(s)$ we get

$$|g_k(s)| \ll \exp(-k\sigma \log k + (k^{1-\sigma} - 1)/(1-\sigma) + O(k)), \tag{3.14}$$

uniformly in k and s, for $|s-1| = 1/10\sqrt{k}$, $\mathrm{Re}(s) < 1$, and since $\sigma = 1 + O(1/\sqrt{k})$ this simplifies to

$$|g_k(s)| \ll \exp(-k \log k + O(k)) \tag{3.15}$$

on this arc. For $k = 0$ or 1 we simply take $R = 1$ and conclude $|g_0(s)| \ll 1$. Rather than make $k = 0$ or 1 explicitly exceptional cases, we ask that the reader substitute this bound for that of (3.12) whenever reference is made to (3.12) with an index of 0 or 1.

The contribution to $E_k(x)$ from $|t| \geqslant T$ is thus, by (3.12), bounded by

$$|E_k(x)|_{\text{wings}} \ll x^2 e^{O(k)} \int_T^\infty (1/t^2) \sum_{j=0}^{k} \frac{1}{j!} \left(\frac{\log t}{\log\log t}\right)^j \cdot \exp(-(k-j)\log(k-j) + (k-j)\log\log(k-j+2))\, dt. \tag{3.16}$$

For $j \geqslant k/2$, we can dispense with the last factor and bound the contribution from these terms by

$$x^2 e^{O(k)} \int_T^\infty (1/t^2) \sum_{j \geqslant k/2} (1/j!)(\log t/\log\log t)^j\, dt \ll x^2 e^{O(k)} \cdot \sum_{j \geqslant k/2} \left(1/j! \int_{\log T}^\infty e^{-u}(u/\log u)^j\, du\right). \tag{3.17}$$

Now

$$\int_{\log T}^\infty e^{-u}(u/\log u)^j\, du \ll \exp(j\log j - j\log\log j + O(j)), \tag{3.18}$$

because for $u \leqslant j/2$, the integrand is increasing rapidly so that relatively little of the mass is found in that part, while for $u > j/2$, $e^{-u}(u/\log u)^j \ll \exp(-j\log\log j + O(j))\, u^j e^{-u}$, and $\int_0^\infty u^j e^{-u}\, du = j! = \exp(j\log j + O(j))$. Thus the quantity in (3.17) is

$$\begin{aligned} &\ll x^2 e^{O(k)} \sum_{j \geqslant k/2} (1/j!) \exp(j\log j - j\log\log j + O(j)) \\ &= x^2 \exp(-\tfrac{1}{2}k\log\log k + O(k)). \end{aligned} \tag{3.19}$$

On the other hand, that part of (3.16) due to $j < k/2$ amounts to $\ll x^2 \exp(-k\log\log k + O(k))$, as we now prove.

For $j < k/2$, $\exp(-(k-j)\log(k-j) + (k-j)\log\log(k-j)) \leqslant \exp(-(k-j)\log k + (k-j)\log\log k + O(k))$. Thus that part of (3.16) due to small j is

$$\ll x^2 \exp(-k\log k + k\log\log k + O(k)) \sum_{j=0}^{[k/2]} (1/j!)(k/\log k)^j \cdot \int_T^\infty (1/t^2)(\log t/\log\log t)^j\, dt. \tag{3.20}$$

The integral in (3.20) is $\ll j!\exp(-j\log\log(j+2)+O(j))$, as we have seen, so the sum in (3.20) is

$$\ll \sum_{j=0}^{[k/2]} (k/\log k\log(j+2))^j \ll \exp(k\log k - 2\,k\log\log k),$$

so that the whole of (3.20) is $\ll x^2\exp(-k\log\log k+O(k))$, as claimed. Combined with the estimate (3.19), this shows that

$$E_k(x)_{\text{wings}} \ll x^2\exp(-\tfrac{1}{2}k\log\log k+O(k)). \tag{3.21}$$

Remark. With a bit more care, we could drop the $\frac{1}{2}$ here, but sharper estimates in (3.21) do not improve the final result.

On the central segment of $\mathscr{C}'$, $|t|\leqslant T$ and so $|t|\ll\sqrt{k}$. Thus the contribution to $E_k(x)$ from the middle part of $\mathscr{C}'$ satisfies

$$\begin{aligned}|E_k(x)|_{\text{mid}} &\ll x^2e^{O(k)}\sum_{j=2}^{k}\frac{(\sqrt{k})^j}{j!}\exp(-(k-j)\log(k-j)\\&\qquad + (k-j)\log\log(k-j))+O(1)\}\\&\ll x^2e^{O(k)}\sum_{j=0}^{k}((\sqrt{k})^j/(j!(k-j)!))\,e^{(k-j)\log\log k}\\&\ll x^2\exp(-\tfrac{1}{2}k\log k+k\log\log k+O(k)),\end{aligned} \tag{3.22}$$

which is less than the bound in (3.21). This completes the proof of Lemma 2 for case $\nu=\omega$.

4. Interpolation

Both for $\nu=\Omega$ and $\nu=\omega$, we now have

$$M(x,k)=\tfrac{1}{2}x^2(\log x)^{k-1}\,h(1,u)(1+O_C(k/\log x))/k!(k-1)!. \tag{4.1}$$

Now let

$$m(x,k)=\tfrac{1}{2}x^2(\log x)^{k-1}\,h(1,u)/k!(k-1)!,$$

and $n(x,k)=dm/dx$.

Since $N(x,k)$ is non-decreasing, if $h>0$ then

$$hN(x,k)\leqslant M(x+h,k)-M(x,k). \tag{4.2}$$

Now

$$\begin{aligned} M(x+h, k) - M(x, k) &\leqslant m(x+h, k) - m(x, k) + O\left(\frac{k}{\log x} m(x, k)\right) \\ &\leqslant hn(x, k) + O(h^2(\log x)^{k-1}/k!(k-1)!) \\ &\quad + O\left(\frac{k}{\log x} m(x, k)\right) \end{aligned}$$

by Taylor's theorem.

With $h = x\sqrt{k/\log x}$, we see that

$$N(x, k) \leqslant n(x, k)(1 + O(\sqrt{1/\log x})). \tag{4.3}$$

By a similar consideration of $M(x, k) - M(x-h, k)$, we get a like lower bound, so

$$N(x, k) = n(x, k)(1 + O(\sqrt{k/\log x})). \tag{4.4}$$

Since $n(x, k) = (x(\log x)^{k-1}/k!(k-1)!)\, h(1, u)(1 + O(k/\log x))$, it only remains to show that $h(1, u) = \Gamma(2-u)$ or $1/\Gamma(1+u)$, depending on the case in question. From (2.1) and (2.4), in case $v = \Omega$,

$$\begin{aligned} h(1, z) &= e^{(\gamma-1)z} \prod_{n=2}^{\infty} \left(1 - \frac{z}{n}\right)^{-1} e^{-z/n} \\ &= e^{\gamma z}(1-z) \prod_{n=1}^{\infty} (1 - z/n)^{-1} e^{-z/n} \\ &= (1-z)\, \Gamma(1-z) = \Gamma(2-z), \end{aligned}$$

by the well-known product formula for the gamma function. When $v = \omega$,

$$\begin{aligned} h(1, z) &= e^{(\gamma-1)z} \prod_{n=2}^{\infty} (1 + z/(n-1))\, e^{-z/n} \\ &= e^{\gamma z} \prod_{n=2}^{\infty} (1 + z/(n-1))\, e^{-z/(n-1)} = 1/\Gamma(1+z) \end{aligned}$$

by the same formula.

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